Spike-Vortex Solutions for Nonlinear Schrödinger System

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Spike-Vortex Solutions for Nonlinear Schrödinger System

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We study the asymptotic behavior of Spike-Vortex solutions of the Nonlinear Schrödinger System :

$$\begin{cases} \varepsilon^2\Delta u - u + u^3 + \beta u|v|^2 = 0 \quad \text{in } B \subset \mathbb{R}^2, \\ \varepsilon^2\Delta v + v - |v|^2 v + \beta u^2 v = 0 \quad \text{in } B \subset \mathbb{R}^2, \\ u = 0 \quad \text{on } \partial B, \\ v = e^{\sqrt{-1}\theta} \quad on \ \partial B, \end{cases}$$

where $u(x) > 0$, $v(x) \in \mathbb{C}$, ε and β are positive real scalars, $B = \left\{x \middle| \, |x| < 1\right\}$.

We construct Spike-Vortex solutions for the above system which have k spikes ($k \in \mathbb{N}$ and $k \geq 2$) in u component and a vortex in v component, and investigate the interaction of the spikes and vortex, where the parameter β plays an important role. In a Spike-Vortex Solution constructed above, the vortex is located at the center while the spikes are radically symmetrically located around the vortex but close to the boundary.

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非線性薛定諤方程组的 spike-vortex 解

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摘要

本論文研究以下非線性薛定諤(Schrödinger)方程组的 spike-vortex 解的漸進行為，

$$\begin{cases} \varepsilon^2\Delta u - u + u^3 + \beta u|v|^2 = 0 & in\ B \subset R^2, \\ \varepsilon^2\Delta v + v - |v|^2 v + \beta u^2 v = 0 & in\ B \subset R^2, \\ u = 0 & on\ \partial B, \\ v = e^{\sqrt{-1}\theta} & on\ \partial B, \end{cases}$$

其中 $u(x)$ 是正实值函数， $v(x)$ 是复值函数， ε 和 β 都是很小的正實數， $B = \{x \| x| < 1\}$。

對于上面的方程組，我們構造了一系列 Spike-Vortex 解，這些解在 u 這個部分含有 k ($k \in N, k > 2$) 個 Spikes，在 v 這個部分含有一個 Vortex。我們還研究 Spikes 與 Vortex 之間的相互作用，發現 β 在其中起重要的作用。具體來說，在一個我們前面所構造的解中，Vortex 位于單位球的中心，Spikes 徑向對稱的位于 Vortex 的週圍，並且非常靠近邊界。

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Chapter 1

Introduction

Spikes and vortices are very important phenomena in nonlinear Schrödinger equations (NLSE) having applications in many physical problems. Especially in Bose-Einstein condensation, they are useful to get skyrmions which have important applications in nuclear physics (cf. [16]) and analogous structures are postulated for early universe cosmology (cf. [3]). For more information about skyrmions, we can refer to [2], [17], [11]. In the last two decades, there have been many analytical works on both spikes and vortices, respectively. One may refer to [18] for a good survey on spikes and [1], [8], [21] for survey on vortices. Recently, a double condensate i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine states has been observed and described by two-component systems of NLSE (cf. [22]). It is then possible to find spike and vortex solutions from two-component systems of NLSE.

In [15], spike-vortex solutions are found for the NLSE systems in $\mathbb{R}^2$:

$$\begin{cases} \triangle u - u + u^3 + \beta u|v|^2 = 0 \quad \text{in } \mathbb{R}^2, \\ \triangle v + v - |v|^2 v + \beta u^2 v = 0 \quad \text{in } \mathbb{R}^2, \\ u > 0, \quad v \in \mathbb{C}. \end{cases} \tag{1.1}$$

For $\beta > 0$ sufficiently small, the spike-vortex solutions exhibiting k spikes in u component and a vortex in v component are found. The vortex is located at the original point and the spikes are symmetrically located around but far away from the vortex. The effect of β to the spikes and vortices is that the positive sign of β may contribute to inter-component repulsion between spikes and vortices which balances with self-attraction in u component. Contrarily, for $\beta < 0$, the radially symmetric spike-vortex solutions with one spike and one vortex both located at the original point are found. In [15], half-skyrmions are also constructed from these spike-vortex solutions produced in the above two situations. (The inter-component interaction between spikes is studied in [13] and [14]; especially in [13], the positive sign of β may give inter-component attraction while the negative sign of β may give inter-component repulsion. For further reference of physical explain, we can go to [28], [29], [30], [31])

In this paper, we study the asymptotic behavior of Spike-Vortex solutions of a similar system in a unit ball :

$$\begin{cases} \varepsilon^2 \triangle u - u + u^3 + \beta u|v|^2 = 0 \quad \text{in } B \subset \mathbb{R}^2, \\ \varepsilon^2 \triangle v + v - |v|^2 v + \beta u^2 v = 0 \quad \text{in } B \subset \mathbb{R}^2, \\ u = 0 \quad \text{on } \partial B, \\ v = e^{\sqrt{-1}\,\theta} \quad \text{on } \partial B, \end{cases} \tag{1.2}$$

where $u > 0$, $v \in \mathbb{C}$, $\beta > 0$ and $B = \left\{x \middle| \ |x| < 1\right\}$. In addition, we investigate the interaction of spikes and vortices.

For functions u , v defined in the unit ball B, we set

$$u_\varepsilon(x) = u(\varepsilon x), \quad v_\varepsilon(x) = v(\varepsilon x) \quad \text{in } B_\varepsilon \subset \mathbb{R}^2, \tag{1.3}$$

where $B_\varepsilon = \left\{x \middle| \ |x| < \frac{1}{\varepsilon}\right\}$. Then (u, v) solves the equation (1.2) if and only if $(u_\varepsilon, v_\varepsilon)$ solves :

$$\begin{cases} \triangle u_\varepsilon - u_\varepsilon + u_\varepsilon^3 + \beta u_\varepsilon |v_\varepsilon|^2 = 0, \quad u_\varepsilon > 0 \quad \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ \triangle v_\varepsilon + v_\varepsilon - |v_\varepsilon|^2 v_\varepsilon + \beta u_\varepsilon^2 v_\varepsilon = 0, \quad v_\varepsilon \in \mathbb{C} \quad \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ u_\varepsilon = 0, \quad v_\varepsilon = e^{\sqrt{-1}\theta} \quad \text{on } \partial B_\varepsilon. \end{cases} \tag{1.4}$$

If $\beta = 0$, (1.4) turns into two independent equations :

$$\begin{cases} \Delta u_\varepsilon - u_\varepsilon + u_\varepsilon^3 = 0, \quad u_\varepsilon > 0 \quad \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ u_\varepsilon = 0 \quad \text{on } \partial B_\varepsilon, \end{cases} \tag{1.5}$$

$$\begin{cases} \Delta v_\varepsilon + v_\varepsilon\Big(1 - |v_\varepsilon|^2\Big) = 0, \quad v_\varepsilon \in \mathbb{C} \quad \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ v_\varepsilon = e^{\sqrt{-1}\theta} \quad \text{on } \partial B_\varepsilon, \end{cases} \tag{1.6}$$

where the first equation, i.e (1.5), can produce spikes and the second one, i.e (1.6), can produce vortices. It is well known that the following elliptic equation :

$$\Delta u - u + u^3 = 0, \quad u > 0, \quad u \in H^1(\mathbb{R}^2) \tag{1.7}$$

has a unique solution $w = w(r)$ for $r = |x|$ with

$$w(r) = A_0 r^{-\frac{1}{2}} e^{-r}\Big[1 + O\Big(\frac{1}{r}\Big)\Big], \quad w'(r) = -A_0 r^{-\frac{1}{2}} e^{-r}\Big[1 + O\Big(\frac{1}{r}\Big)\Big], \tag{1.8}$$

where A_0 is a positive constant. (1.6) is the well-known Ginzburg-Landau equation (cf. [1], [21], [23]) having a radically symmetric vortex solution of the form :

$$v_0(x) = S_\varepsilon(r)\, e^{\sqrt{-1}\,\theta}, \quad r = |x|, \tag{1.9}$$

where S_ε satisfies

$$\begin{cases} S_\varepsilon'' + \frac{1}{r} S_\varepsilon' - \frac{1}{r^2} S_\varepsilon + S_\varepsilon - S_\varepsilon^3 = 0, \\ S_\varepsilon(0) = 0, S_\varepsilon\left(\frac{1}{\varepsilon}\right) = 1, \end{cases} \tag{1.10}$$

and

$$S_\varepsilon > 0, \quad S_\varepsilon(r) = 1 - \frac{1}{2r^2} + O\Big(\frac{1}{r^4}\Big) \quad \text{as } r \gg 1. \tag{1.11}$$

In this paper, we want to prove that when ε and β are sufficiently small (β depends on ε and $\beta \to 0$ as $\varepsilon \to 0$), there exist spikes-vortex solutions for the system (1.2). Before stating the main result, we modify w and define another function w_β, where

$$w_\beta(x) := \sqrt{1-\beta}\, w(\sqrt{1-\beta}\, x).$$

Our main theorem is in the following :

THEOREM 1.1. *For $k \geq 2$, $k \in \mathbb{N}$, ε small enough, $Ce^{-(\frac{1}{\varepsilon})^q} \leq \beta \leq C\varepsilon^{4+3\alpha}$, where $0 < q < 1, 0 < \alpha < \frac{1}{2}$, system (1.2) has a solution (u, v). Moreover, as $\varepsilon \to 0_+$, (u, v) has the following asymptotic form*

$$\begin{cases} u(z) = \sum_{j=1}^{k} w_\beta(\frac{z}{\varepsilon} - \xi_j^\varepsilon) + o(1), \\ v(z) = S_\varepsilon(\frac{r}{\varepsilon})e^{\sqrt{-1}\,\theta + o(1)}, \end{cases} \tag{1.12}$$

where $\langle \xi_1^\varepsilon, ..., \xi_k^\varepsilon \rangle$ forms a regular k-polygon with

$$\xi_j^\varepsilon = le^{\sqrt{-1}\frac{2\pi(j-1)}{k}}, \; j = 1, ..., k \tag{1.13}$$

(we can consider a point (x, y) in $\mathbb{R}^2$ to be a complex value number $x + \sqrt{-1}y$).

Throughout the paper, we choose l such that

$$l \in \Lambda = \left[\frac{1}{\varepsilon} - \gamma_1 \ln \frac{1}{\varepsilon}, \frac{1}{\varepsilon} - \gamma_2 \ln \frac{1}{\varepsilon}\right], \tag{1.14}$$

where γ_1 and γ_2 are two positive constants to be determined later. Then we can conclude that

$$1 \ll \left(\frac{1}{\varepsilon} - l\right) \ll \frac{1}{\varepsilon}, \quad \forall l \in \Lambda, \tag{1.15}$$

which is a very useful condition for we to analyze the properties of spike.

We prove the theorem by using the so-called *Localized Energy Method*, or *LEM* in short, which is generally used to find solutions with concentrating behavior (cf. [27], [7]). As in [15], we consider function spaces with some symmetry of rotation and conjugation (defined by (2.16)). In these spaces, the kernel of the linearized operator of spike or vortex respectly will both be simpler (see Lemma 2.3), and the solutions are also of this symmetry. The rest of this paper is organized as follows. In chapter 2, we will define an approximate solution of equation system (1.4) and get some useful properties of the approximate solution. In chapter 3, we use Liapunov-Schmidt reduction process to find spike-vortex solutions. In chapter 4, in virtue of a reduction lemma, solving (1.2) is equivalent to finding a critical point l of an energy functional (defined by (4.1)), which is an one-dimensional problem; we find a local minimizer of the energy and estimate the location of the minimizer (it determines the location of the spikes).

In closing this chapter, we remark that throughout the whole paper, unless otherwise stated, the letter C will always denote various positive generic constants, which are independent on all parameters.

Chapter 2

Properties of approximate solutions

Recall that

$$w_\beta(x) := \sqrt{1-\beta}\, w(\sqrt{1-\beta}x), \tag{2.1}$$

where w satisfies the equation :

$$\Delta w - w + w^3 = 0, \quad w > 0 \quad \text{in } \mathbb{R}^2.$$

We see that w_β satisfies

$$\Delta w_\beta - (1-\beta) w_\beta + w_\beta^3 = 0, \quad w_\beta > 0 \quad \text{in } \mathbb{R}^2. \tag{2.2}$$

It is easy to check

$$w_\beta \to w \text{ uniformly in } C^1(\mathbb{R}^2), \quad \text{as} \quad \beta \to 0 \tag{2.3}$$

and as in (1.8),

$$w_\beta(r) = [A_0 + o(1)]\, r^{-\frac{1}{2}} e^{-\sqrt{1-\beta}r} \Big[1 + O\Big(\frac{1}{r}\Big)\Big], \tag{2.4}$$

$$w'_\beta(r) = -\,[A_0 + o(1)]\, r^{-\frac{1}{2}} e^{-\sqrt{1-\beta}r} \Big[1 + O\Big(\frac{1}{r}\Big)\Big]. \tag{2.5}$$

We assume that the location of $k-$spikes is characterized by

$$\xi_j^\varepsilon = l e^{\sqrt{-1}\frac{2\pi(j-1)}{k}},\ j = 1, ..., k$$

and define the approximate solution $(u_0,\ v_0)$ in the following :

$$\begin{cases} u_0(x) = \sum_{j=1}^{k} P_{\xi_j^\varepsilon} w(x), \\ v_0(x) = S_\varepsilon(r) e^{\sqrt{-1}\theta}, r = |x|, \end{cases} \tag{2.6}$$

where $P_{\xi_j^\varepsilon} w$ is the unique solution of

$$\begin{cases} \triangle u(x) - (1-\beta)u(x) + \big(w_\beta(x - \xi_j^\varepsilon)\big)^3 = 0 \quad \text{ in } B_\varepsilon \subset \mathbb{R}^2, \\ u(x) = 0 \quad \text{on } \partial B_\varepsilon. \end{cases} \tag{2.7}$$

In this paper, we need to use some properties of $P_{\xi_j^\varepsilon} w$, which mainly come from Proposition 3.1 in [5]. For more information about $P_{\xi_j^\varepsilon} w$, we can also refer to [7], [10], [19]. Using comparison principle, we get the estimate of $P_{\xi_j^\varepsilon} w$,

$$0 < P_{\xi_j^\varepsilon} w < w_\beta(x - \xi_j^\varepsilon) < C e^{-\sqrt{1-\beta}\, r}, \quad r = |x - \xi_j^\varepsilon|. \tag{2.8}$$

Regularity theory implies that w_β and also $P_{\xi_j^\varepsilon} w$ are smooth, and hence

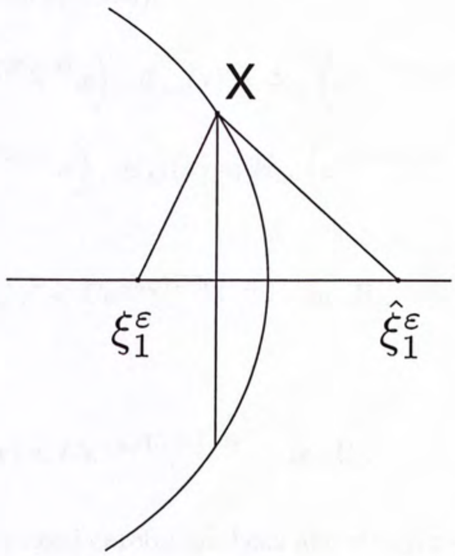
$$\triangle \frac{\partial P_{\xi_j^\varepsilon} w}{\partial l} - (1-\beta)\frac{\partial P_{\xi_j^\varepsilon} w}{\partial l} + 3\frac{\partial w_\beta(x - \xi_j^\varepsilon)}{\partial l}\, w_\beta(x - \xi_j^\varepsilon)^2 = 0 \quad \text{ in } B_\varepsilon. \tag{2.9}$$

We let $\hat{\xi}_j^\varepsilon = (\frac{2}{\varepsilon} - l)\, e^{\sqrt{-1}\ \frac{2\pi(j-1)}{k}}$ (i.e. $\hat{\xi}_j^\varepsilon$ is the symmetric point of ξ_j^ε with respect to the boundary, as in the figure, we draw a local figure near ξ_1^ε where an arbitrary boundary point is denoted by X), and set

$$\Phi_{i,\varepsilon}(x) = w_\beta(x - \xi_i^\varepsilon) - P_{\xi_i^\varepsilon} w(x).$$

$\Phi_{i,\varepsilon}(x)$ satisfies the following equation :

$$\begin{cases} \triangle \Phi_{i,\varepsilon}(x) - (1-\beta)\Phi_{i,\varepsilon}(x) = 0 \quad \text{ in } B_\varepsilon \subset \mathbb{R}^2, \\ \Phi_{i,\varepsilon}(x) = w_\beta(x - \xi_i^\varepsilon) \quad \text{on } \partial B_\varepsilon. \end{cases}$$



The Maximum Principle implies that

$$0 < \Phi_{i,\varepsilon}(x) < \sup_{x \in \partial B_\varepsilon} w_\beta(x - \xi_i^\varepsilon) < Ce^{-\sqrt{1-\beta}\,(\frac{1}{\varepsilon}-l)} \quad \text{in } B_\varepsilon \subset \mathbb{R}^2. \tag{2.10}$$

We decompose $\Phi_{i,\varepsilon}(x)$ into three parts as follows,

$$\Phi_{i,\varepsilon}(x) = w_\beta(x - \hat{\xi}_i^\varepsilon) + \Psi_{i,\varepsilon}(x) + K_{i,\varepsilon}(x), \tag{2.11}$$

where $\Psi_{i,\varepsilon}(x)$ and $K_{i,\varepsilon}(x)$ satisfy the following equations :

$$\begin{cases} \triangle \Psi_{i,\varepsilon}(x) - (1-\beta)\Psi_{i,\varepsilon}(x) = 0 & \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ \Psi_{i,\varepsilon}(x) = w_\beta(x - \xi_i^\varepsilon) - w_\beta(x - \hat{\xi}_i^\varepsilon) & \text{on } \partial B_\varepsilon, \end{cases}$$

$$\begin{cases} \triangle K_{i,\varepsilon}(x) - (1-\beta)K_{i,\varepsilon}(x) = w_\beta(x - \hat{\xi}_i^\varepsilon)^3 & \text{in } B_\varepsilon \subset \mathbb{R}^2, \\ K_{i,\varepsilon}(x) = 0 \quad \text{on } \partial B_\varepsilon. \end{cases}$$

Because w_β is of radically symmetric and $\xi_i^\varepsilon = e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}\xi_j^\varepsilon$, it follows that

$$w_\beta(x - \xi_j^\varepsilon) = w_\beta\left(e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}x - \xi_i^\varepsilon\right).$$

Then it is evident that the families $\{P_{\xi_i^\varepsilon}w\}$, $\{\Phi_{i,\varepsilon}\}$, $\{\Psi_{i,\varepsilon}\}$ and $\{K_{i,\varepsilon}\}$ are all of some symmetry of rotation. More precisely,

$$P_{\xi_j^\varepsilon}w(x) = P_{\xi_i^\varepsilon}w\left(e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}x\right),\ \Phi_{j,\varepsilon}(x) = \Phi_{i,\varepsilon}\left(e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}x\right),$$

$$\Psi_{j,\varepsilon}(x) = \Psi_{i,\varepsilon}\left(e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}x\right),\ K_{j,\varepsilon}(x) = K_{i,\varepsilon}\left(e^{\sqrt{-1}\frac{2\pi(i-j)}{k}}x\right).$$

Since

$$0 < w_\beta(x-\hat{\xi}_i^\varepsilon)^3 < Ce^{-3\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} \quad \text{in } B_\varepsilon,$$

the comparison principle gives

$$0 < -K_{i,\varepsilon}(x) < Ce^{-3\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} \quad \text{in } B_\varepsilon. \tag{2.12}$$

To get estimate of $\Psi_{i,\varepsilon}(x)$, we need careful analysis about $w_\beta(x-\xi_i^\varepsilon)-w_\beta(x-\hat{\xi}_i^\varepsilon)$ on the boundary. More precisely, we prove the following lemma.

Lemma 2.1. *When $l \in \Lambda$, we have*

$$0 < \Psi_{i,\varepsilon}(x) < o(1)\left[e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + w_\beta(x-\hat{\xi}_i^\varepsilon)\right] \quad \textit{in } B_\varepsilon. \tag{2.13}$$

Proof. Because of the symmetry of rotation, we only need to do analysis for $i=1$. To apply the comparison principle, we need the estimate of the boundary data. Since $l \in \Lambda$, we have $1 \ll \left(\frac{1}{\varepsilon}-l\right) \ll \frac{1}{\varepsilon}$, which is used frequently in the following analysis. Let us take $x=(x_1,x_2) \in \partial B_\varepsilon$, it follows that $x_1^2+x_2^2 = \left(\frac{1}{\varepsilon}\right)^2$. We distinguish two cases :

Case 1 : $\frac{1}{\varepsilon}-x_1 \geq 4\left[\varepsilon(\frac{1}{\varepsilon}-l)\right]^{\frac{1}{2}}$.

As $\xi_1^\varepsilon = (l, 0)$ (or $\xi_1^\varepsilon = l$ as a complex value number), we have

$$\begin{aligned}
|x-\xi_1^\varepsilon|^2 &= |x_2|^2 + |x_1 - l|^2 \\
&= (\frac{1}{\varepsilon} - l)^2 + 2(\frac{1}{\varepsilon} - x_1)l \\
&\geq (\frac{1}{\varepsilon} - l)^2 + 4\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{\frac{1}{2}}\frac{1}{\varepsilon} \\
&\geq (\frac{1}{\varepsilon} - l)^2 + 2(\frac{1}{\varepsilon} - l)\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{2}} + 2(\frac{1}{\varepsilon} - l)\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{2}} \\
&\geq (\frac{1}{\varepsilon} - l)^2 + 2(\frac{1}{\varepsilon} - l)\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{4}} + \Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{2}} \\
&= \left\{\frac{1}{\varepsilon} - l + \Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{4}}\right\}^2.
\end{aligned}$$

Consequently, we get

$$|x-\xi_1^\varepsilon| \geq (\frac{1}{\varepsilon} - l) + [\varepsilon(\frac{1}{\varepsilon} - l)]^{-\frac{1}{4}}.$$

Using above inequality and the facts that

$$\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{-\frac{1}{4}} \gg 1,$$

we have

$$0 < w_\beta(x-\xi_1^\varepsilon) \leq o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}.$$

Besides, it is easily to see for x on the boundary of B_ε,

$$|x-\xi_1^\varepsilon| \leq |x-\hat{\xi}_1^\varepsilon|,$$

then

$$w_\beta(x-\hat{\xi}_1^\varepsilon) \leq w_\beta(x-\xi_1^\varepsilon).$$

All together, we have

$$0 \leq w_\beta(x-\xi_1^\varepsilon) - w_\beta(x-\hat{\xi}_1^\varepsilon) \leq 2w_\beta(x-\xi_1^\varepsilon) \leq o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}.$$

Case 2 : $\frac{1}{\varepsilon} - x_1 < 4\Big[\varepsilon(\frac{1}{\varepsilon} - l)\Big]^{\frac{1}{2}}$.

In this case, as $\varepsilon(\frac{1}{\varepsilon} - l) \ll 1$, we have

$$\big|\, |x - \xi_1^\varepsilon| - |x - \hat{\xi}_1^\varepsilon| \,\big| \le 2(\frac{1}{\varepsilon} - x_1) = o(1)$$

and

$$1 \ll (\frac{1}{\varepsilon} - l) \le |x - \xi_1^\varepsilon| \le |x - \hat{\xi}_1^\varepsilon|,$$

whence

$$\begin{aligned}
& w_\beta(x - \xi_1^\varepsilon) - w_\beta(x - \hat{\xi}_1^\varepsilon) \\
&= w_\beta(x - \hat{\xi}_1^\varepsilon)\left(\frac{w_\beta(x - \xi_1^\varepsilon)}{w_\beta(x - \hat{\xi}_1^\varepsilon)} - 1\right) \\
&= w_\beta(x - \hat{\xi}_1^\varepsilon)\left[\left(1 + \frac{|x - \xi_1^\varepsilon| - |x - \hat{\xi}_1^\varepsilon|}{|x - \hat{\xi}_1^\varepsilon|}\right)^{-\frac{1}{2}} e^{\sqrt{1-\beta}\left(|x - \hat{\xi}_1^\varepsilon| - |x - \xi_1^\varepsilon|\right)} - 1 + o(1)\right] \\
&= w_\beta(x - \hat{\xi}_1^\varepsilon)\Big[\,(1 + o(1))\, e^{\sqrt{1-\beta}\left(|x - \hat{\xi}_1^\varepsilon| - |x - \xi_1^\varepsilon|\right)} - 1 + o(1)\Big] \\
&= o(1) w_\beta(x - \hat{\xi}_1^\varepsilon).
\end{aligned}$$

The combination of the above two cases gives us for $x \in \partial B_\varepsilon$

$$0 \le w_\beta(x - \xi_i^\varepsilon) - w_\beta(x - \hat{\xi}_i^\varepsilon) \le o(1)\Big[e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon} - l)} + w_\beta(x - \hat{\xi}_i^\varepsilon)\Big].$$

Using comparison principle, we get

$$0 < \Psi_{i,\varepsilon}(x) < o(1)\Big[e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon} - l)} + w_\beta(x - \hat{\xi}_i^\varepsilon)\Big] \quad \text{in } B_\varepsilon.$$

□

Then Corollary 3.1 of [5] can be proved in the same way. We state it in the following.

COROLLARY 2.2. *Setting $P_{\xi_j^\varepsilon} w(x) = 0$ for $x \notin B_\varepsilon$, we have*

$$P_{\xi_j^\varepsilon} w(x) \to w_\beta(x - \xi_j^\varepsilon) \quad \text{in } H^1(\mathbb{R}^2) \text{ and } L^\infty(\mathbb{R}^2)$$

as $\frac{1}{\varepsilon} - l \to +\infty$ uniformly for $l < \frac{1}{\varepsilon}$. □

To study the spike-vortex solution, we use a key transformation here. We set

$$u(x) = u_0(x) + \phi(x)\ ,\ v = v_0(x)e^{\sqrt{-1}\psi(x)} \text{ and } \ \psi(x) = \psi_1(x) + \sqrt{-1}\psi_2(x),$$

where u_0 and v_0 are the approximate solutions defined by (2.6), $\psi_1(x),\ \psi_2(x) \in \mathbb{R}$. Then, solving (1.4) is equivalent to solving :

$$\begin{cases} \Delta\phi - \phi + 3u_0^2\phi = \\ \quad -\Big\{\phi^3 + 3\phi^2 u_0 + \Delta u_0 - u_0 + u_0^3 + \beta(u_0+\phi)S_\varepsilon^2 e^{-2\psi_2}\Big\} \qquad \text{in } B_\varepsilon, \\ \Delta\psi + \frac{2\nabla v_0}{v_0}\nabla\psi - 2\sqrt{-1}S_\varepsilon^2\psi_2 = \\ \quad \sqrt{-1}S_\varepsilon^2(1 - e^{-2\psi_2} - 2\psi_2) - \sqrt{-1}(\nabla\psi)^2 + \sqrt{-1}\beta(u_0+\phi)^2 \quad \text{in } B_\varepsilon, \\ \phi = 0\ ,\ \psi = 0 \quad \text{on } \partial B_\varepsilon. \end{cases} \tag{2.14}$$

We define three operators $L_1,\ L_2$ and $\hat{L}_2$ in the following :

$$\begin{cases} L_1[\phi] = \Delta\phi - \phi + 3u_0^2\phi, \\ L_2[\psi] = \Delta\psi + \frac{2\nabla v_0}{v_0}\nabla\psi - 2\sqrt{-1}S_\varepsilon^2\psi_2, \\ \hat{L}_2[\varphi] = \Delta\varphi + \varphi - S_\varepsilon^2\varphi - 2Re\Big(S_\varepsilon(r)e^{-\sqrt{-1}\theta}\varphi\Big)S_\varepsilon(r)e^{\sqrt{-1}\theta}, \end{cases}$$

where S_ε is defined by (1.10) and of the property (1.11), $\varphi = -\sqrt{-1}v_0\psi$. We also denote :

$$\begin{cases} N_1[\phi,\psi] = -\Big(\phi^3 + 3\phi^2 u_0 + \beta\phi S_\varepsilon^2 e^{-2\psi_2} + \beta u_0 S_\varepsilon^2(e^{-2\psi_2} - 1)\Big), \\ S_1[u_0,v_0] = -(\Delta u_0 - u_0 + u_0^3 + \beta u_0 S_\varepsilon^2), \\ N_2[\phi,\psi] = \sqrt{-1}\Big[S_\varepsilon^2(1 - e^{-2\psi_2} - 2\psi_2) - (\nabla\psi)^2 + \beta(\phi^2 + 2u_0\phi)\Big], \\ S_2[u_0,v_0] = \sqrt{-1}\beta u_0^2. \end{cases} \tag{2.15}$$

Then, (2.14) is equivalent to the following equations system :

$$\begin{cases} L_1[\phi] = S_1[u_0,v_0] + N_1[\phi,\psi], \\ L_2[\psi] = S_2[u_0,v_0] + N_2[\phi,\psi], \\ \hat{L}_2[\varphi] = -\sqrt{-1}v_0\Big(S_2[u_0,v_0] + N_2[\phi,\psi]\Big). \\ \phi = 0\ ,\ \psi = 0\ ,\ \varphi = 0 \quad \text{on } \partial B_\varepsilon. \end{cases}$$

We define symmetric function spaces just like in [15],

$$\Sigma = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R} \times \mathbb{C} \,\middle|\, \begin{matrix} u \in \Sigma_1 \\ v \in \Sigma_2 \end{matrix} \right\}, \tag{2.16}$$

and

$$\Sigma_1 = \left\{ u \middle| u(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) = u(z), \quad u(\bar{z}) = u(z) \right\},$$
$$\Sigma_2 = \left\{ v \middle| v(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) = e^{\sqrt{-1}\,\frac{2\pi}{k}} v(z), \quad v(\bar{z}) = v(z)^* \right\}.$$

Hereafter, both the over-bar and asterisk denote complex conjugate. We remark that the equation (1.4) is invariant under the following two maps

$$\begin{cases} T_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) \\ e^{-\sqrt{-1}\,\frac{2\pi}{k}} v(ze^{\sqrt{-1}\,\frac{2\pi}{k}}) \end{pmatrix}, \\ \\ T_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(\bar{z}) \\ v(\bar{z})^* \end{pmatrix}. \end{cases}$$

Therefore, we may look for solutions of (1.4) in the space Σ. The lemma proved in [15] and stated below expresses the kernel of the operators L_1 and $\hat{L}_2$ on the symmetric function spaces.

LEMMA 2.3.

(1) Suppose $L_1[\phi] = 0$, $\phi \in H^2(\mathbb{R}^2)$ and $\phi(\bar{z}) = \phi(z)$. Then $\phi(z) = C\frac{\partial w}{\partial z_1}(z)$, where $z = z_1 + \sqrt{-1}\, z_2$, $z_j \in \mathbb{R}$ and C is a constant.

(2) Suppose

$$\hat{L}_2[\psi] = 0, \ |\psi| \le C, \ \psi(z)^* = \psi(\bar{z}) \ \text{and} \ \psi(ze^{\sqrt{-1}\frac{2\pi}{k}}) = e^{\sqrt{-1}\frac{2\pi}{k}}\psi(z),$$

where C is a positive constant. Then $\psi \equiv 0$.

For arbitrary fixed constant α and γ with $0 < \alpha,\ \gamma < 1,$ we introduce some weighted function spaces and norms as follows

$$\begin{aligned} X &= \Big\{\psi \mid \|\psi\|_* < +\infty\Big\}, \\ Y &= \Big\{h \mid \|h\|_{**} < +\infty\Big\}, \end{aligned}$$

with associated norm defined by

$$\begin{aligned} \|\psi\|_* &= \Big\| |\psi_1| + |x||\nabla\psi_1| + |x|^{1+\alpha}|\psi_2| + |x|^{1+\alpha}|\nabla\psi_2| \Big\|_{L^\infty(|x|>2)} + \Big\|\psi\Big\|_{C^{1,\gamma}(|x|<2)}, \\ \|h\|_{**} &= \Big\| |x|^{2+\alpha}|h_1| + |x|^{1+\alpha}|h_2| \Big\|_{L^\infty(|x|>2)} + \Big\|v_0 h\Big\|_{C^{0,\gamma}(|x|<3)}. \end{aligned}$$

where $h = h_1 + \sqrt{-1}h_2,\ h_j \in \mathbb{R}$.

For further application, we derive estimates of error of the approximate solution in the following lemma :

LEMMA 2.4. *For $l \in \Lambda$, we have*

$$\Big\|S_1[u_0, v_0]\Big\|_{L^2(B_\varepsilon)} \le C\Big(\frac{\beta}{l^2} + o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + e^{-2l\sin\frac{\pi}{k}}\Big), \tag{2.17}$$

$$\Big\|S_2[u_0, v_0]\Big\|_{**} \le C\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}. \tag{2.18}$$

PROOF. Before starting the proof, we introduce some notations by denoting

$$I := \{(i,j,m) \mid i,j,m = 1,\cdots,k\} - \{(i,i,i) \mid i = 1,\cdots,k\},$$

and

$$w_j(x) := w_\beta(x - \xi_j^\varepsilon).$$

Since

$$u_0(x) = \sum_{j=1}^k P_{\xi_j^\varepsilon} w(x) = \sum_{j=1}^k \Big[w(x - \xi_j^\varepsilon) - \Phi_{i,\varepsilon}(x)\Big],$$

we have

$$\begin{aligned} S_1[u_0, v_0] &= -(\triangle u_0 - u_0 + u_0^3 + \beta u_0 S_\varepsilon^2) \\ &= -\sum_{j=1}^{k} \Big[\triangle P w_j - (1-\beta) P w_j + w_j^3\Big] \\ &\quad -(u_0^3 - \sum_{j=1}^{k} w_j^3) - \beta u_0 (S_\varepsilon^2 - 1) \\ &= -(u_0^3 - \sum_{j=1}^{k} w_j^3) - \beta u_0 (S_\varepsilon^2 - 1). \end{aligned} \tag{2.19}$$

We begin with the first term in (2.19),

$$\begin{aligned} u_0^3 - \sum_{j=1}^{k} w_j^3 &= \sum_{(i,j,m)\in I} [w_i w_j w_m - 3 w_i w_j \Phi_{m,\varepsilon} + 3 w_i \Phi_{j,\varepsilon}\Phi_{m,\varepsilon} - \Phi_{i,\varepsilon}\Phi_{j,\varepsilon}\Phi_{m,\varepsilon}] \\ &\quad - \sum_{i=1}^{k} [3 w_i^2 \Phi_{i,\varepsilon} - 3 w_i \Phi_{i,\varepsilon}^2 + \Phi_{i,\varepsilon}^3]. \end{aligned} \tag{2.20}$$

We get the estimate of every component of (2.20) in the following.

From equality (2.4), we have

$$|w_i^2 w_j| \le e^{-2\sqrt{1-\beta}|x-\xi_i^\varepsilon|} e^{-\sqrt{1-\beta}|x-\xi_j^\varepsilon|}. \tag{2.21}$$

It is easy to conclude that

$$\begin{aligned} &\left\| e^{-2\sqrt{1-\beta}|x-\xi_i^\varepsilon|} e^{-\sqrt{1-\beta}|x-\xi_j^\varepsilon|} \right\|_{L^2(B_\varepsilon)} \\ &\le C \sup_{x\in B_\varepsilon} e^{-\sqrt{1-\beta}(|x-\xi_j^\varepsilon|+|x-\xi_i^\varepsilon|)} \| e^{-\sqrt{1-\beta}|x-\xi_i^\varepsilon|} \|_{L^2(B_\varepsilon)} \end{aligned} \tag{2.22}$$

and

$$\sup_{x\in B_\varepsilon} e^{-\sqrt{1-\beta}(|x-\xi_j^\varepsilon|+|x-\xi_i^\varepsilon|)} \le e^{-\sqrt{1-\beta}(|\xi_j^\varepsilon-\xi_i^\varepsilon|)} \le e^{-2\sqrt{1-\beta}\, l \sin\frac{\pi}{k}}. \tag{2.23}$$

By combining (2.21), (2.22) and (2.23), we obtain

$$\|w_i^2 w_j\|_{L^2(B_\varepsilon)} \le e^{-2l\sin\frac{\pi}{k}}. \tag{2.24}$$

From equality (2.4), equation (2.11), inequality (2.12) and lemma 2.1, we have

$$\|w_i^2 \Phi_{i,\varepsilon}\|_{L^2(B_\varepsilon)} \le o(1) e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}. \tag{2.25}$$

Easily, we can find

$$\Big\|\sum_{(i,j,m)\in I} w_i w_j w_m\Big\|_{L^2(B_\varepsilon)} = O\Big(\|w_i^2 w_j\|_{L^2(B_\varepsilon)}\Big), \tag{2.26}$$

and

$$\Big\|\sum_{i,j,m=1}^{k}[-w_i w_j \Phi_{m,\varepsilon} + w_i \Phi_{j,\varepsilon}\Phi_{m,\varepsilon} - \Phi_{i,\varepsilon}\Phi_{j,\varepsilon}\Phi_{m,\varepsilon}]\Big\|_{L^2(B_\varepsilon)} = O\Big(\|w_i^2 \Phi_{i,\varepsilon}\|_{L^2(B_\varepsilon)}\Big). \tag{2.27}$$

Hence we get the estimate of (2.20) in $\|\cdot\|_{L^2(B_\varepsilon)}$ norm in virtue of (2.24), (2.25), (2.26) and (2.27),

$$\Big\|u_0^3 - \sum_{j=1}^{k} w_j^3\Big\|_{L^2(B_\varepsilon)} \le C\Big(o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + e^{-2l\sin\frac{\pi}{k}}\Big). \tag{2.28}$$

Because of equality (1.11) and inequality (2.8), we get the estimate of the second term in (2.19) directly,

$$\Big\|\beta u_0(S_\varepsilon^2 - 1)\Big\|_{L^2(B_\varepsilon)} \le C\frac{\beta}{l^2}. \tag{2.29}$$

Finally, the estimate in (2.17) can be easily derived from(2.28) and (2.29).

For $S_2[u_0, v_0]$ part, we compute directly as follows,

$$\begin{aligned}
\Big\|S_2[u_0,v_0]\Big\|_{**} &\le \beta\left(\Big\||x|^{1+\alpha}u_0^2\Big\|_{L^\infty(|x|>2)} + \Big\|v_0 u_0^2\Big\|_{C^{0,\gamma}(|x|<3)}\right)\\
&\le \beta\left(\Big\|\sum_{i,j=1}^{k}|x|^{1+\alpha}P_{\xi_i^\varepsilon}w_\beta P_{\xi_j^\varepsilon}w_\beta\Big\|_{L^\infty(|x|>2)} + C\right)\\
&\le \beta\left(\Big\|\sum_{i,j=1}^{k}|x|^{1+\alpha}e^{-(|x-\xi_i^\varepsilon|+|x-\xi_j^\varepsilon|)}\Big\|_{L^\infty(|x|>2)} + C\right)\\
&\le C\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}.
\end{aligned}$$

□

Chapter 3

Liapunov-Schmidt Reduction

In this chapter, we consider the invertibility of the linearized operators L_1 and L_2 in some proper function spaces.

LEMMA 3.1. *Suppose ε is small enough and $l \in \Lambda$, if $\phi \in H_0^2(B_\varepsilon)\bigcap\Sigma_1$ and $h \in L^2(B_\varepsilon)\bigcap\Sigma_1$ satisfy*

$$\begin{cases} \Delta\phi - \phi + 3\phi u_0^2 = h + c(l)\frac{\partial u_0}{\partial l} \quad \text{in } B_\varepsilon, \\ \phi = 0 \quad \text{on } \partial B_\varepsilon, \\ \int_{B_\varepsilon} \phi\frac{\partial u_0}{\partial l} = 0, \end{cases} \tag{3.1}$$

then

$$\|\phi\|_{H^2(B_\varepsilon)} \leq C\|h\|_{L^2(B_\varepsilon)}. \tag{3.2}$$

PROOF. By contradiction, we assume that there exist ε_n, β_n, l_n, $c(l_n) \in \mathbb{R}$, $\phi_n \in H_0^2(B_\varepsilon)\bigcap\Sigma_1$, $h_n \in L^2(B_\varepsilon)\bigcap\Sigma_1$ satisfy (3.1) and

$$\varepsilon_n,\ \beta_n \to 0, \quad l_n \in \Lambda_n, \quad \|\phi_n\|_{H^2(B_\varepsilon)} = 1 \quad \text{and} \quad \|h_n\|_{L^2(B_\varepsilon)} \to 0, \quad \text{as} \quad n \to \infty,$$

where

$$\Lambda_n = \left[\frac{1}{\varepsilon_n} - \gamma_1 \ln\frac{1}{\varepsilon_n}, \frac{1}{\varepsilon_n} - \gamma_2 \ln\frac{1}{\varepsilon_n}\right] \tag{3.3}$$

and γ_1 and γ_1 are two positive constants to be determined in the last chapter .

To lead to a contradiction argument, we divide the proof into 3 steps.

step 1 : In this step, we shall show that

$$c(l_n) \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

By equation (3.1) and the assumption, integrating by part, we have

$$\begin{aligned} c(l_n)\int_{B_\varepsilon}(\frac{\partial u_0}{\partial l})^2 &= \int_{B_\varepsilon}(\triangle\phi_n - \phi_n + 3\phi_n u_0^2)\frac{\partial u_0}{\partial l} - \int_{B_\varepsilon} h_n\frac{\partial u_0}{\partial l} \\ &= \int_{B_\varepsilon}(\triangle\frac{\partial u_0}{\partial l} - \frac{\partial u_0}{\partial l} + 3\frac{\partial u_0}{\partial l}u_0^2)\phi_n + o(1) \\ &= \sum_{i=1}^k\int_{B_\varepsilon}(\triangle\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} - \frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} + 3\frac{\partial w_i}{\partial l}w_i^2)\phi_n \\ &\quad + \sum_{i=1}^k\int_{B_\varepsilon}(3\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}u_0^2 - 3\frac{\partial w_i}{\partial l}w_i^2)\phi_n + o(1). \end{aligned} \tag{3.5}$$

Since $\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}$ satisfies equation (2.9), immediately we have

$$\sum_{i=1}^k\int_{B_\varepsilon}(\triangle\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} - \frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} + 3\frac{\partial w_i}{\partial l}w_i^2)\phi_n = -\beta_n\sum_{i=1}^k\int_{B_\varepsilon}\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}\phi_n.$$

Corollary 2.2 says $P_{\xi_i^\varepsilon}w(x) \to w_i$ in $H^1(\mathbb{R}^2)$, together with the assumption $\|\phi_n\|_{H^2} = 1$, we derive

$$\beta_n\sum_{i=1}^k\int_{B_\varepsilon}\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}\phi_n = \beta_n\sum_{i=1}^k\int_{B_\varepsilon}\frac{\partial w_i}{\partial l}\phi_n + o(1) \le C\beta_n.$$

Hence we may conclude

$$\sum_{i=1}^k\int_{B_\varepsilon}(\triangle\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} - \frac{\partial P_{\xi_i^\varepsilon}w}{\partial l} + 3\frac{\partial w_i}{\partial l}w_i^2)\phi_n = o(1). \tag{3.6}$$

Corollary 2.2 also implies $\Phi_{i,\varepsilon_n} \to 0$ in $H^1(B_\varepsilon)$, as $n \to \infty$. Accordingly, we have

$$\begin{aligned} &\sum_{i=1}^k\int_{B_\varepsilon}(3\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}u_0^2 - 3\frac{\partial w_i}{\partial l}w_i^2)\phi_n \\ =\ &\sum_{i=1}^k\int_{B_\varepsilon}(3\frac{\partial w_i}{\partial l}(u_0^2 - w_i^2) - 3\frac{\partial \Phi_{i,\varepsilon_n}}{\partial l}u_0^2)\phi_n \\ =\ &\sum_{i=1}^k\int_{B_\varepsilon}\Big[-3\frac{\partial w_i}{\partial l}(P_{\xi_i^\varepsilon}w + w_i)\Phi_{i,\varepsilon_n} + \sum_{(j,m)\neq(i,i)}3\frac{\partial w_i}{\partial l}P_{\xi_j^\varepsilon}wP_{\xi_m^\varepsilon}w - 3\frac{\partial \Phi_{i,\varepsilon_n}}{\partial l}u_0^2\Big]\phi_n \\ =\ &o(1). \end{aligned} \tag{3.7}$$

But for some positive constant c_0 independent on n, when n large enough,

$$\int_{B_\varepsilon} (\frac{\partial u_0}{\partial l})^2 = \sum_{i=1}^{k} \int_{B_\varepsilon} (\frac{\partial P_{\xi_i^\varepsilon} w}{\partial l})^2 + o(1) > c_0 > 0. \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into (3.5), we get

$$c(l_n) \to 0 \quad \text{as } n \to \infty.$$

step 2 : In this step, we shall prove, in a subsequence

$$\phi_{i,n} \rightharpoonup 0 \in H^2(\mathbb{R}^2),\ \forall i = 1, \ldots, k.$$

We begin this step by introducing some notations and intermediate functions. Let $\chi_n(x)$ be a smooth cut-off function such that

$$\chi_n(x) = \begin{cases} 1, & \text{for } |x| \le \frac{1}{2}(\frac{1}{\varepsilon_n} - l), \\ 0, & \text{for } |x| \ge \frac{1}{\varepsilon_n} - l, \end{cases}$$

and define

$$\phi_{i,n}(x) = \chi_n(x)\phi_n(x + \xi_{i,n}),$$

then we can extend $\phi_{i,n}$ to the whole space $\mathbb{R}^2$, provided that we set $\phi_{i,n}(x) = 0$ for $|x| \ge \frac{1}{\varepsilon_n} - l$. Since

$$\|\phi_{i,n}\|_{H^2(\mathbb{R}^2)} \le C\|\phi_n\|_{H^2(B_\varepsilon)},$$

there exist $\phi_{i,\infty} \in H^2(\mathbb{R}^2)$ and a subsequence of $\{\phi_{i,n}\}$ (we also denote it to be $\{\phi_{i,n}\}$), such that

$$\phi_{i,n} \rightharpoonup \phi_{i,\infty} \quad \text{in } H^2(\mathbb{R}^2). \tag{3.9}$$

In $B_n = \left\{|x| < \frac{1}{2}(\frac{1}{\varepsilon_n} - l)\right\}$, $\phi_{i,n}$ satisfies

$$\begin{aligned} &\Delta\phi_{i,n} - \phi_{i,n} + 3\phi_{i,n}w^2 + 3\phi_{i,n}\left(P_{\xi_i^\varepsilon} w^2(x - \xi_{i,n}) - w^2\right) \\ &+3\phi_{i,n} \sum_{(j,m)\neq(i,i)} P_{\xi_j^\varepsilon} w(x - \xi_{i,n}) P_{\xi_m^\varepsilon} w(x - \xi_{i,n}) \\ &= h_n(x - \xi_{i,n}) + c(l_n)\frac{\partial u_0}{\partial l}(x - \xi_{i,n}). \end{aligned}$$

By using Corollary 2.2, estimates (3.4), (2.8), the facts $\beta_n \to 0$ and $\|h_n\|_{L^2(B_\varepsilon)} \to 0$, it is easily derived that $\phi_{i,\infty}$ satisfies the equation

$$\Delta\phi_{i,\infty} - \phi_{i,\infty} + 3w^2\phi_{i,\infty} = 0. \tag{3.10}$$

When $i = 1$, $\xi_{1,n} = l_n$ as a complex number. Since $\phi_n \in \Sigma_1$, we have

$$\phi_{1,n}(\bar{x}) = \chi_n(\bar{x})\phi_n(\bar{x} + \xi_{1,n}) = \chi_n(\bar{x})\phi_n(x + \bar{\xi}_{1,n}) = \phi_{1,n}(x). \tag{3.11}$$

Sobolev embedding implies that $\phi_{1,n} \to \phi_{1,\infty}$ in C^0, so equality (3.11) yields

$$\phi_{1,\infty}(\bar{x}_0) = \phi_{1,\infty}(x_0) \quad \forall \quad x_0. \tag{3.12}$$

With (3.10) and (3.12), Lemma 2.3 tells us for some constant c,

$$\phi_{1,\infty} = c\frac{\partial w}{\partial x_1}. \tag{3.13}$$

We want to show in (3.13) $c = 0$ by the orthogonal condition,

$$\int_{B_\varepsilon} \phi_n \frac{\partial u_0}{\partial l} = 0, \ \forall n.$$

We have

$$\int_{B_\varepsilon} \phi_n \frac{\partial u_0}{\partial l} = \sum_{i=1}^{k} \int_{B_\varepsilon} \chi_{i,n}\phi_n \frac{\partial u_0}{\partial l} + \int_{B_\varepsilon} \Big(1 - \sum_{i=1}^{k} \chi_{i,n}\Big)\phi_n \frac{\partial u_0}{\partial l}. \tag{3.14}$$

Since

$$\begin{aligned} \Big|\int_{B_{\varepsilon_n}} \Big(1 - \sum_{i=1}^{k} \chi_{i,n}\Big)\phi_n \frac{\partial u_0}{\partial l}\Big| \ &\le \ \sum_{i=1}^{k} \int_{B_{\varepsilon_n} - B_{(\xi_{i,n}, \frac{1}{2}(\frac{1}{\varepsilon_n} - l))}} \Big|\phi_n \frac{\partial P_{\xi_i^\varepsilon} w}{\partial l}\Big| \\ &\le \ \sum_{i=1}^{k} \int_{B_{\varepsilon_n} - B_{(\xi_{i,n}, \frac{1}{2}(\frac{1}{\varepsilon_n} - l))}} \Big|\phi_n \frac{\partial w_i}{\partial l}\Big| + \Big|\phi_n \frac{\partial \Phi_{i,\varepsilon_n}}{\partial l}\Big|, \end{aligned}$$

Corollary 2.2 and formular (2.5) imply that

$$\lim_{n\to\infty} \int_{B_{\varepsilon_n}} \Big(1 - \sum_{i=1}^{k} \chi_{i,n}\Big)\phi_n \frac{\partial u_0}{\partial l} = 0. \tag{3.15}$$

We also use Corollary 2.2 and formular (2.5) to obtain

$$\lim_{n\to\infty}\int_{B_{\varepsilon_n}}\chi_{j,n}\phi_n\frac{\partial P_{\xi_i^\varepsilon}w}{\partial l}=\lim_{n\to\infty}\int_{B_{\varepsilon_n}}\chi_{j,n}\phi_n\Big(\frac{\partial w_i}{\partial l}-\frac{\partial \Phi_{i,\varepsilon_n}}{\partial l}\Big)=0\quad \forall j\neq i. \tag{3.16}$$

By the symmetry of ϕ_n, $\frac{\partial u_0}{\partial l}$, $\forall i,j$, we have

$$\int_{B_\varepsilon}\chi_{i,n}\phi_n\frac{\partial u_0}{\partial l}=\int_{B_\varepsilon}\chi_{j,n}\phi_n\frac{\partial u_0}{\partial l}. \tag{3.17}$$

Substituting (3.15), (3.16), (3.17) into (3.14), in virtue of the orthogonal condition, we discover

$$\lim_{n\to\infty}\int_{B_\varepsilon}\chi_{1,n}\phi_n\frac{\partial u_0}{\partial l}=0. \tag{3.18}$$

On the other hand, we have

$$\begin{aligned}\int_{B_\varepsilon}\chi_{1,n}\phi_n\frac{\partial u_0}{\partial l} &= \int_{B_\varepsilon}\chi_{1,n}\phi_n\Big(\frac{\partial w_1}{\partial l}-\frac{\partial \Phi_{1,\varepsilon_n}}{\partial l}\Big)+\int_{B_\varepsilon}\chi_{1,n}\phi_n\sum_{j\neq 1}\frac{\partial P_{\xi_j^\varepsilon}w}{\partial l}\\ &= \int_{\mathbb{R}^2}\phi_{1,n}\frac{\partial w_\beta}{\partial x_1}+o(1).\end{aligned} \tag{3.19}$$

By (3.18) and (3.19), it is clear

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\phi_{1,n}\frac{\partial w_\beta}{\partial x_1}=0,$$

then together with (2.3) and (3.9), we can conclude

$$\int_{\mathbb{R}^2}\phi_{1,\infty}\frac{\partial w}{\partial x_1}=0.$$

Finally, from (3.13) and the last formula, we derive

$$\phi_{1,\infty}\equiv 0,$$

which implies $\phi_{1,n}\rightharpoonup 0\in H^2(\mathbb{R}^2)$. The symmetry tells us

$$\phi_{i,n}\rightharpoonup 0\in H^2(\mathbb{R}^2),\ \forall i=1,\dots,k.$$

step 3 : In this step, we shall prove

$$\|\phi_n\|_{H^2(B_\varepsilon)}\to 0.$$

The process comes mainly from [7]. For any fixed length R, denoting $B_R = \Big\{x\Big||x| < R\Big\}$, there holds

$$\|\phi_{i,n}\|_{H^2(B_R)} \leq \|\phi_{i,n}\|_{H^2(\mathbb{R}^2)} \leq C.$$

Then we have compact embedding in a subsequence,

$$\exists \phi_{i,R},\ s.t. \quad \phi_{i,n} \to \phi_{i,R} \in C^0(B_R).$$

The fact $\phi_{i,n} \rightharpoonup 0 \in H^2(\mathbb{R}^2)$ proved in step 2 implies $\phi_{i,R} = 0$, so

$$\phi_{i,n} \to 0 \ \ in \ \ L^2(B_R). \tag{3.20}$$

From (2.8), we know that

$$\|3u_0^2\phi_n\|_{L^2(B_R^c)} \leq Ce^{-2(1-\beta)R},$$

where $B_R^c = B_\varepsilon - \bigcup_{i=1}^{k} B_{(\xi_{i,n},R)}$.

Hence together with (3.20), for n large enough, we have

$$\|3u_0^2\phi_n\|_{L^2(B_\varepsilon)} \leq Ce^{-2(1-\beta)R}.$$

Choosing R arbitrary large, we can conclude

$$\|3u_0^2\phi_n\|_{L^2(B_\varepsilon)} \to 0. \tag{3.21}$$

Besides, since $c(l_n) \to 0$ and $\|h_n\|_{L^2(B_\varepsilon)} \to 0$, it follows

$$\Big\|h_n + c(l_n)\frac{\partial u_0}{\partial l}\Big\|_{L^2(B_\varepsilon)} \to 0. \tag{3.22}$$

Furthermore, by using the first equation in (3.1), estimates (3.21) and (3.22), we can get

$$\Big\|(\triangle - 1)\phi_n\Big\|_{L^2(B_\varepsilon)} \to 0,$$

which gives $\|\phi_n\|_{H^1(B_\varepsilon)} \to 0$. By L^p estimate, we conclude $\|\phi_n\|_{H^2(B_\varepsilon)} \to 0$, which contradicts the assumption $\|\phi_n\|_{H^2(B_\varepsilon)} = 1$. Thus the estimate (3.2) holds and we have finished the proof of this lemma. □

Now we have got the prior estimate for linearized operator L_1 in the above lemma and we shall use the process in Lemma 4.1 of [23] to get the prior estimate for linearized operator L_2 in the following lemma.

LEMMA 3.2. *There exists a constant $C > 0$, $\forall\varepsilon$ small enough, if $\psi \in H_0^1(B_\varepsilon)\bigcap X$, $-\sqrt{-1}v_0\psi \in \Sigma_2$, $h \in L^2(B_\varepsilon)\bigcap Y$, $-\sqrt{-1}v_0 h \in \Sigma_2$ satisfies*

$$\begin{cases} L_2\psi = h & in\ B_\varepsilon, \\ \psi = 0 & on\ \partial B_\varepsilon, \end{cases} \tag{3.23}$$

we have

$$\|\psi\|_\star \le C\|h\|_{\star\star}.$$

PROOF. We prove by contradiction. Let us assume that there exists a sequence of $\varepsilon_n \in \mathbb{R}$, $\psi^n \in H_0^1(B_\varepsilon)\bigcap X$, $-\sqrt{-1}v_0\psi^n \in \Sigma_2$, $h^n \in L^2(B_\varepsilon)\bigcap Y$, $-\sqrt{-1}v_0 h^n \in \Sigma_2$ satisfies (3.23) with $\|\psi^n\|_\star = 1$, $\varepsilon_n \to 0$, $\|h^n\|_{\star\star} \to 0$.

In the following, we divide the proof into three steps. Since step 1 and step 2 are just the same as the proof of lemma 4.1 in [23], I just state the result without details.

step 1 : For arbitrary fixed constant $0 < \delta < \frac{1}{2}$, in the region $\frac{\delta}{\varepsilon_n} < |x| < \frac{1}{\varepsilon_n}$, we can get

$$|\psi_1^n| + \varepsilon_n^{-1}|\nabla\psi_1^n| \to 0, \tag{3.24}$$

$$\varepsilon_n^{-(1+\alpha)}|\psi_2^n + \nabla\psi_2^n| \to 0. \tag{3.25}$$

step 2 : Denote

$$\eta_n(x) = \begin{cases} 1, & \text{for } |x| \le \frac{\delta}{\varepsilon_n}, \\ 0, & \text{for } |x| \ge \frac{2\delta}{\varepsilon_n}, \end{cases} \tag{3.26}$$

and set $\hat{\psi}^n = \eta_n\psi^n$. For any fixed R_0 large enough (independent on n), we have

$$|\hat{\psi}_1^n| + |r\nabla\hat{\psi}_1^n| \le C\Big[\|\hat{\psi}^n\|_{C^1(1<r<R_0)} + o(1)\Big] \quad \text{for } r > 2, \tag{3.27}$$

$$|\hat{\psi_2^n}| + |\nabla \hat{\psi_2^n}| \leq \frac{C}{r^{(1+\alpha)}}\left[\|\hat{\psi^n}\|_{C^1(1<r<R_0)} + o(1)\right] \quad \text{for } r > 2, \tag{3.28}$$

where $r = |x|$.

step 3 : Set $\varphi^n = -\sqrt{-1}v_0\psi^n$, then we get

$$\nabla\varphi^n = -\sqrt{-1}(\nabla v_0\psi^n + \nabla\psi^n v_0)$$

and

$$\hat{L}_2\varphi^n = -\sqrt{-1}v_0 h^n.$$

The L^p theory gives

$$\begin{aligned}\|\varphi^n\|_{H^{2,q}(B_R)} &\leq c_1(R)\left(\|\varphi^n\|_{L^q(B_{2R})} + \|-\sqrt{-1}v_0h^n\|_{L^q(B_{2R})}\right)\\ &\leq c_2(R)\left(\|\psi^n\|_\star + \|h^n\|_{\star\star}\right),\end{aligned} \tag{3.29}$$

where $c_1(R)$ and $c_1(R)$ are constants only dependent on the radius R. Then Sobolev embedding theorem implies that there is a subsequence denoted also by φ^n convergent uniformly to φ_R^0 in $C^{1,\gamma}(B_R)$. By a diagonal process, we get some φ^0 such that $\varphi^0 = \varphi_R^0$ for $|x| < R$. Hence φ^n, passing to a subsequence (denoted also by φ^n), converges to some φ^0, locally uniformly in $C^{1,\gamma}$, which solves

$$\hat{L}_2\varphi^0 = 0 \quad in\ \mathbb{R}^2. \tag{3.30}$$

$\varphi^n \in \Sigma_2$, then we get $\varphi^0 \in \Sigma_2$. Because

$$\|\varphi^0\|_{L^\infty(B_R)} = \lim_{n\to\infty}\|\varphi^n\|_{L^\infty(B_R)} \leq C\lim_{n\to\infty}\|\psi^n\|_\star = C$$

holds for arbitrary R, we have

$$\|\varphi^0\|_{L^\infty(\mathbb{R}^2)} \leq 1.$$

Then Lemma 2.3 implies

$$\varphi^0 = 0.$$

Therefore we get

$$\varphi^n \to 0 \quad in\ C^{1,\gamma}(B_{R_0}), \tag{3.31}$$

which produce

$$\|\psi^n\|_{C^{1,\gamma}(1<|x|<R_0)} \to 0 \quad as\ n \to \infty. \tag{3.32}$$

Since

$$-\sqrt{-1}v_0 h^n(x) \in C^{0,\gamma}(|x|<3),$$

by Schauder's theory, we derive

$$\|\varphi^n\|_{C^{2,\gamma}(|x|<2)} \le C\left(\|\varphi^n\|_{L^\infty(|x|<3)} + \|-\sqrt{-1}v_0 h^n\|_{C^{0,\gamma}(|x|<3)}\right). \tag{3.33}$$

Besides, it is easy to conclude

$$\|\psi^n\|_{C^{1,\gamma}(|x|<2)} \le C\|\varphi^n\|_{C^{2,\gamma}(|x|<2)}. \tag{3.34}$$

By combining (3.31), (3.33), (3.34) and the assumption $\|h^n\|_{**} \to 0$, we get

$$\|\psi^n\|_{C^{1,\gamma}(|x|<2)} \to 0,\ as\ n \to \infty. \tag{3.35}$$

Finally the combination of estimates (3.24), (3.25), (3.27), (3.28) (3.32) and (3.35) produce $\|\psi^n\|_\star \to 0, \quad as\ n \to \infty$, which contradicts to our assumption at the beginning. □

After we obtain the prior estimate, the existence and uniqueness in the linear theory may follow from the standard Fredholm's alternatives. By a contraction mapping argument, we can solve the projected problem as follows.

Lemma 3.3. *$\forall\varepsilon$ small enough, $l \in \Lambda$ and $\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha} \ll 1$, the problem*

$$\begin{cases} L_1[\phi] = S_1[u_0,v_0] + N_1[\phi,\psi] + c(l)\frac{\partial u_0}{\partial l} & in\ B_\varepsilon \\ L_2[\psi] = S_2[u_0,v_0] + N_2[\phi,\psi] & in\ B_\varepsilon \\ \phi = 0 & in\ \partial B_\varepsilon \\ \psi = 0 & in\ \partial B_\varepsilon \\ \int_{B_\varepsilon} \phi\frac{\partial u_0}{\partial l} = 0 \end{cases} \tag{3.36}$$

has a solution (ϕ_l, ψ_l) *with* $\phi \in H_0^2(B_\varepsilon)\bigcap \Sigma_1,\ \ \psi \in H_0^1(B_\varepsilon)\bigcap X, -\sqrt{-1}v_0\psi \in \Sigma_2$ *and*

$$\begin{aligned} \|\phi_l\|_{H^2(B_\varepsilon)} &\le C\Big(\frac{\beta}{l^2} + o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + e^{-2l\sin\frac{\pi}{k}}\Big), && (3.37)\\ \|\psi_l\|_\star &\le C\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}. && (3.38) \end{aligned}$$

Proof. For the purpose of applying contraction map principle, we define

$$\mathcal{B} = \left\{(\phi,\psi):\quad \begin{aligned} &\phi \in \Sigma_1,\ -\sqrt{-1}v_0\psi \in \Sigma_2,\\ &\|\phi\|_{H^2(B_\varepsilon)} \le C\Big(\frac{\beta}{l^2} + o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + e^{-2l\sin\frac{\pi}{k}}\Big),\\ &\|\psi\|_\star \le C\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha} \end{aligned}\right\}.$$

Because $\forall\varepsilon$ small enough, $l \in \Lambda$ and $\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha} \ll 1$, it is easy to conclude

$$\|\phi_l\|_{H^2(B_\varepsilon)} \ll 1 \text{ and } \|\psi_l\|_\star \ll 1. \tag{3.39}$$

Through simple computation, we get

$$\Big\|N_1[\phi,\psi]\Big\|_{L^2(B_\varepsilon)} \le C\Big(\|\phi\|^2_{H^2(B_\varepsilon)} + \|\phi\|^3_{H^2(B_\varepsilon)} + \beta\|\phi\|_{H^2(B_\varepsilon)} + \beta\|\psi\|_\star\Big) \tag{3.40}$$

and

$$\Big\|N_2[\phi,\psi]\Big\|_{\star\star} \le C\left(\|\psi\|_\star^2 + \beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}\|\phi\|_{H^2(B_\varepsilon)} + \beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}\|\phi\|^2_{H^2(B_\varepsilon)}\right) \tag{3.41}$$

For any $(\phi_1,\psi_1),\ (\phi_2,\psi_2) \in \mathcal{B}$ with $\psi_1 = \psi_{11} + \sqrt{-1}\psi_{12}$ and $\psi_2 = \psi_{21} + \sqrt{-1}\psi_{22}$,

$$\begin{aligned} N_1[\phi_2,\psi_2] - N_1[\phi_1,\psi_1] &= \beta S_\varepsilon^2(e^{-2\psi_{12}} - e^{-2\psi_{22}})(\phi_2 + u_0)\\ &\quad + (\phi_1 - \phi_2)\Big[3u_0(\phi_1+\phi_2) + \phi_1^2 + \phi_1\phi_2 + \phi_2^2 + \beta S_\varepsilon^2 e^{-2\psi_{12}}\Big] \end{aligned}$$

and

$$\begin{aligned} N_2[\phi_2,\psi_2] - N_2[\phi_1,\psi_1] &= -\sqrt{-1}\beta(\phi_1-\phi_2)(2u_0+\phi_1+\phi_2)\\ &\quad + \sqrt{-1}\Big((\nabla\psi_1)^2 - (\nabla\psi_2)^2\Big) + \sqrt{-1}S_\varepsilon^2(e^{-2\psi_{12}} - e^{-2\psi_{22}} + 2\psi_{12} - 2\psi_{22}). \end{aligned}$$

Then we have

$$\begin{aligned}\Big\|N_1[\phi_1,\psi_1]-N_1[\phi_2,\psi_2]\Big\|_{L^2(B_\varepsilon)} \leq&\\ &C\|\phi_1-\phi_2\|_{L^2(B_\varepsilon)}(\|\phi_1\|_{H^2(B_\varepsilon)}+\|\phi_2\|_{H^2(B_\varepsilon)}+\beta)\\ &+C\beta\|\psi_1-\psi_2\|_\star(C+\|\phi_2\|_{H^2(B_\varepsilon)}) \qquad (3.42)\end{aligned}$$

and

$$\begin{aligned}\Big\|N_2[\phi_1,\psi_1]-N_2[\phi_2,\psi_2]\Big\|_{\star\star} \leq&\\ &C\beta\left(\frac{1}{\varepsilon}\right)^{1+\alpha}\|\phi_1-\phi_2\|_{H^2(B_\varepsilon)}(\|\phi_1\|_{H^2(B_\varepsilon)}+\|\phi_2\|_{H^2(B_\varepsilon)}+C)\\ &+C\|\psi_1-\psi_2\|_\star(\|\psi_1\|_\star+\|\psi_2\|_\star). \qquad (3.43)\end{aligned}$$

Finally, we get the desired result by contraction mapping principle. Here lemmas 3.1, 3.2 and the estimates above are used. □

We get the C^1 regularity of the map $l \mapsto (c(l)\ ,\ \phi_l\ ,\ \psi_l)$ from lemma (3.3) and the Implicit Function Theorem (see [27], [7], [24], [25] and [26] for detail).

Chapter 4

Critical point of the reduced energy functional

By the result in chapter 3, for all ε small enough, if we find some $l \in \Lambda$ such that $c(l) = 0$, we solve (1.4). In this chapter we shall find the zero of $c(l)$ by searching the critical point of the corresponding energy functional with respect to l.

We define the energy

$$\begin{aligned} E_\varepsilon[u,v] &= \frac{1}{2}\int_{B_\varepsilon} |\nabla u|^2 + \frac{1}{2}\int_{B_\varepsilon} |u|^2 - \frac{1}{4}\int_{B_\varepsilon} |u|^4 \\ &\quad -\frac{\beta}{2}\int_{B_\varepsilon} |uv|^2 + \frac{1}{2}\int_{B_\varepsilon} |\nabla v|^2 + \frac{1}{4}\int_{B_\varepsilon} (1-|v|^2)^2. \end{aligned} \tag{4.1}$$

LEMMA 4.1. *For any (ϕ_l, ψ_l) in Lemma 3.3, letting $u_l = u_0 + \phi_l$, $v_l = v_0 e^{\sqrt{-1}\psi_l}$, if l_ε is a critical point of $E_\varepsilon[u_l, v_l]$, we have $c(l_\varepsilon) = 0$.*

PROOF. If l_ε is a critical point of $E_\varepsilon[u_l, v_l]$, we have

$$\frac{d}{dl} E_\varepsilon[u_l, v_l] \mid_{l_\varepsilon} = 0. \tag{4.2}$$

Equivalently,

$$\int_{B_\varepsilon}\Big[\nabla u_l\nabla\frac{du_l}{dl}+u_l\frac{du_l}{dl}-u_l^3\frac{du_l}{dl}-\beta u_l\frac{du_l}{dl}|v_l|^2$$
$$+\frac{1}{2}\Big(\nabla\bar{v}_l\nabla\frac{dv_l}{dl}-\bar{v}_l(1-|v_l|^2)\frac{dv_l}{dl}-\beta u_l^2\bar{v}_l\frac{dv_l}{dl}$$
$$+\nabla v_l\nabla\frac{d\bar{v}_l}{dl}-v_l(1-|v_l|^2)\frac{d\bar{v}_l}{dl}-\beta u_l^2v_l\frac{d\bar{v}_l}{dl}\Big)\Big]\Big|_{l_\varepsilon}=0. \tag{4.3}$$

Integrating by part, (4.3) turns into

$$\int_{B_\varepsilon}\Big[(\triangle u_l-u_l+u_l^3+\beta u_l|v_l^2|)\frac{du_l}{dl}+\frac{1}{2}(\triangle\bar{v}_l+\bar{v}_l(1-|v_l|^2)+\beta u_l^2\bar{v}_l)\frac{dv_l}{dl}$$
$$+\frac{1}{2}(\triangle v_l+v_l(1-|v_l|^2)+\beta u_l^2v_l)\frac{d\bar{v}_l}{dl}\Big]\Big|_{l_\varepsilon}=0. \tag{4.4}$$

(u_l, v_l) satisfies the equation, so we can simplify (4.4) into

$$\int_{B_\varepsilon}c(l)\frac{du_0}{dl}\frac{du_l}{dl}\Big|_{l_\varepsilon}=0,$$

then

$$c(l_\varepsilon)\int_{B_\varepsilon}\frac{du_0}{dl}\Big(\frac{du_0}{dl}+\frac{d\phi_l}{dl}\Big|_{l_\varepsilon}=0. \tag{4.5}$$

Because

$$\int_{B_\varepsilon}\frac{du_0}{dl}^2 = \sum_{j=1}^k\int_{B_\varepsilon}\frac{dw_j}{dx_1}^2+o(1),$$

and for some independent positive constant C_1

$$\int_{B_\varepsilon}\frac{dw_j}{dx_1}^2>C_1,$$

we have

$$\int_{B_\varepsilon}\frac{du_0}{dl}^2>C_1. \tag{4.6}$$

On the other hand,

$$\int_{B_\varepsilon}\frac{du_0}{dl}\frac{d\phi_l}{dl}<C\|\phi_l\|_{H^2}\le C(\beta+o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+e^{-2l\sin\frac{\pi}{k}})\ll 1. \tag{4.7}$$

Finally, we deduce $c(l)=0$ from (4.5), (4.6), (4.7). $\square$

In the rest of this chapter, we shall find a critical point of the energy functional $E_\varepsilon[u_l, v_l]$. This energy functional can be separated into two parts as follows,

$$E_\varepsilon[u_l, v_l] = E_\varepsilon[u_0 + \phi_l, v_0 e^{\sqrt{-1}\psi_l}] = E_\varepsilon[u_0, v_0] + F[\phi_l, \psi_l],$$

where

$$\begin{aligned} F[\phi_l, \psi_l] &= \int_{B_\varepsilon} \Big[\nabla u_0 \nabla \phi_l + \frac{1}{2}(\nabla \phi_l)^2 + u_0\phi_l + \frac{1}{2}(\phi_l)^2\Big] \\ &\quad - \int_{B_\varepsilon} \Big[u_0\phi_l^3 + \frac{3}{2}u_0^2\phi_l^2 + u_0^3\phi_l + \frac{1}{4}\phi_l^4\Big] \\ &\quad - \beta \int_{B_\varepsilon} \Bigg[\Big(u_0\phi_l + \frac{1}{2}(\phi_l)^2\Big) S_\varepsilon^2 e^{-2\psi_{l2}} + \frac{1}{2}(u_0^2) S_\varepsilon^2 \Big(e^{-2\psi_{l2}} - 1\Big)\Bigg] \\ &\quad + \frac{1}{2}\int_{B_\varepsilon} \Big(|\nabla \hat{\psi}|^2 + \nabla \bar{v}_0 \nabla \hat{\psi} + \nabla v_0 \nabla \bar{\hat{\psi}}\Big) \\ &\quad + \frac{1}{4}\int_{B_\varepsilon} \Bigg[\Big(1 - |v_0 + \hat{\psi}|^2\Big)^2 - \Big(1 - |v_0|^2\Big)^2\Bigg], \end{aligned}$$

$\psi_l = \psi_{l1} + \sqrt{-1}\psi_{l2}$, $\hat{\psi} = v_0\left(e^{\sqrt{-1}\psi_l} - 1\right)$ and $v_l = v_0 + \hat{\psi}$.

We get the estimate of $F[\phi_l, \psi_l]$ in the following lemma, which implies that $F[\phi_l, \psi_l]$ is small enough and not the main part of $E_\varepsilon[u_0 + \phi_l, v_0 e^{\sqrt{-1}\psi_l}]$.

LEMMA 4.2. *For ϕ_l , ψ_l in Lemma 3.3, we have*

$$F[\phi_l, \psi_l] \le C\left[\log\left(\frac{1}{\varepsilon}\right)\beta^2\left(\frac{1}{\varepsilon}\right)^{2+2\alpha} + \left(\beta + o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} + e^{-2l\sin\frac{\pi}{k}}\right)^2\right]. \quad (4.8)$$

PROOF. Since $P_{\xi_i^\varepsilon} w$ satisfies equation(2.7), integrating by part, we get

$$\begin{aligned} \int_{B_\varepsilon} \nabla u_0 \nabla \phi_l &+ u_0\phi_l - u_0^3\phi_l \\ &= \int_{B_\varepsilon} \Big[(\sum_{i=1}^k w_i^3 - u_0^3)\phi_l + \beta u_0 \phi_l\Big] \\ &= \int_{B_\varepsilon} \Big[(\sum_{(i,j,m)\in I} P_{\xi_i^\varepsilon} w P_{\xi_j^\varepsilon} w P_{\xi_m^\varepsilon} w)\phi_l + \sum_{i=1}^k (w_i^3 - P_{\xi_i^\varepsilon}^3)\phi_l + \beta u_0 \phi_l\Big] \\ &\le \Big(\beta + e^{-2l\sin(\frac{\pi}{k})} + o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}\Big)\|\phi_l\|_{H^2(B_\varepsilon)}. \end{aligned} \quad (4.9)$$

$P_{\xi_j^\varepsilon} w$ is of exponent decay and $\|\phi_l\|_{H^2}$, $\|\psi_l\|_\star$ are very small from lemma(3.3), then we have

$$\left|\int_{B_\varepsilon}\Big[\frac{1}{2}(\nabla\phi_l)^2+\frac{1}{2}(\phi_l)^2\Big]-\int_{B_\varepsilon}\Big[u_0\phi_l^3+\frac{3}{2}u_0^2\phi_l^2+\frac{1}{4}\phi_l^4\Big]\right| \leq \|\phi_l\|^2_{H^2(B_\varepsilon)} \tag{4.10}$$

and

$$\left|\beta\int_{B_\varepsilon}\Big(u_0\phi_l+\frac{1}{2}(\phi_l)^2\Big)S_\varepsilon^2 e^{-2\psi_{l2}}\right| \leq \beta\|\phi_l\|_{H^2(B_\varepsilon)}. \tag{4.11}$$

Similarly, in virtue of equation(1.6), and integrating by part, we have

$$\int_{B_\varepsilon}\nabla v_0\nabla\bar{\hat{\psi}}=\int_{B_\varepsilon}\bar{\hat{\psi}}v_0(1-|v_0|^2),$$

and

$$\int_{B_\varepsilon}\nabla\hat{\psi}\nabla\bar{v_0}=\int_{B_\varepsilon}\bar{v_0}\hat{\psi}(1-|v_0|^2).$$

Hence

$$\begin{aligned}&\left|\frac{1}{2}\int_{B_\varepsilon}(|\nabla\hat{\psi}|^2+\nabla\bar{v}_0\nabla\hat{\psi}+\nabla v_0\nabla\bar{\hat{\psi}})+\frac{1}{4}\int_{B_\varepsilon}[(1-|v_0+\hat{\psi}|^2)^2-(1-|v_0|^2)^2]\right|\\ &=\left|\frac{1}{2}\int_{B_\varepsilon}\Big(|\nabla\hat{\psi}|^2-\hat{\psi}^2(1-|v_0|^2)\Big)+\frac{1}{4}\int_{B_\varepsilon}(\hat{\psi}^2+\bar{v}_0\hat{\psi}+v_0\bar{\hat{\psi}})^2\right| \end{aligned}\tag{4.12}$$

Since

$$\hat{\psi}=v_0\left(e^{\sqrt{-1}\psi_l}-1\right)=v_0\left(\sqrt{-1}\psi_l+O(\psi_l^2)\right)$$

and

$$\nabla\hat{\psi}=\nabla v_0\left(e^{\sqrt{-1}\psi_l}-1\right)+\sqrt{-1}\nabla\psi_l v_0 e^{\sqrt{-1}\psi_l},$$

(4.12) implies

$$\begin{aligned}&\left|\frac{1}{2}\int_{B_\varepsilon}(|\nabla\hat{\psi}|^2+\nabla\bar{v}_0\nabla\hat{\psi}+\nabla v_0\nabla\bar{\hat{\psi}})+\frac{1}{4}\int_{B_\varepsilon}[(1-|v_0+\hat{\psi}|^2)^2-(1-|v_0|^2)^2]\right|\\ &\leq\log\left(\frac{1}{\varepsilon}\right)\|\psi\|_\star^2. \end{aligned}\tag{4.13}$$

Besides we have

$$\int_{B_\varepsilon}\beta u_0^2S_\varepsilon^2(e^{-2\psi_{l2}}-1)\leq\beta\|\psi\|_\star. \tag{4.14}$$

All together, we get

$$F[\phi_l,\psi_l] \leq$$
$$C\left[\|\phi_l\|_{H^2}^2+\log\left(\frac{1}{\varepsilon}\right)\|\psi_l\|_\star^2+\beta\|\psi\|_\star+\left(\beta+o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+e^{-2l\sin\frac{\pi}{k}}\right)\|\phi_l\|_{H^2}\right]$$

from (4.9), (4.10), (4.11), (4.13), (4.14). Hence we have proved the lemma if we use the estimates of $\|\phi_l\|_{H^2}$, $\|\psi_l\|_\star$ in lemma(3.3). □

We obtain an expansion of $E_\varepsilon[u_0,v_0]$ in the following form,

$$\begin{aligned} E_\varepsilon[u_0,v_0] &= \frac{1}{2}\int_{B_\varepsilon}|\nabla u_0|^2+\frac{1-\beta}{2}\int_{B_\varepsilon}|u_0|^2-\frac{1}{4}\int_{B_\varepsilon}|u_0|^4 \\ &+ \frac{\beta}{2}\int_{B_\varepsilon}|u_0|^2(1-|v_0|^2)+E[0,v_0]. \end{aligned}$$

From (1.11) and Lemma (2.1), we have for some constant $C_1>0$,

$$\begin{aligned} \frac{\beta}{2}\int_{B_\varepsilon}|u_0|^2(1-|v_0|^2) &= \frac{\beta}{2}\int_{B_\varepsilon}|u_0|^2\left(\frac{1}{r^2}+o(\frac{1}{r^2})\right) \\ &= C_1\frac{\beta}{l^2}\left(1+o(1)\right). \end{aligned} \tag{4.15}$$

[19] claims that for some constant $C_2>0$,

$$\begin{aligned} &\frac{1}{2}\int_{B_\varepsilon}|\nabla u_0|^2+\frac{1-\beta}{2}\int_{B_\varepsilon}|u_0|^2-\frac{1}{4}\int_{B_\varepsilon}|u_0|^4 \\ &= k(1-\beta)^{\frac{3}{2}}I(w)+C_2e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+Ce^{-l\sin(\frac{\pi}{k})}+o\left(e^{-2(\frac{1}{\varepsilon}-l)}+e^{-l\sin(\frac{\pi}{k})}\right) \end{aligned} \tag{4.16}$$

where

$$I(w)=\frac{1}{2}\int_{\mathbb{R}^2}\left(|\nabla w|^2+w^2\right)-\frac{1}{4}\int_{\mathbb{R}^2}w^4.$$

We denote

$$\begin{aligned} G(l) &= F[\phi_l,\psi_l]+\frac{1}{2}\int_{B_\varepsilon}|\nabla u_0|^2+\frac{1-\beta}{2}\int_{B_\varepsilon}|u_0|^2 \\ &-\frac{1}{4}\int_{B_\varepsilon}|u_0|^4+\frac{\beta}{2}\int_{B_\varepsilon}|u_0|^2(1-|v_0|^2)-k(1-\beta)^{\frac{3}{2}}I(w). \end{aligned} \tag{4.17}$$

Substituting (4.8), (4.15), (4.16) into (4.17), we have

$$\begin{aligned} G(l) &= C_1\frac{\beta}{l^2}+C_2e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+Ce^{-l\sin(\frac{\pi}{k})}+o(e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+e^{-l\sin(\frac{\pi}{k})}) \\ &+C[\log\left(\frac{1}{\varepsilon}\right)\beta^2\left(\frac{1}{\varepsilon}\right)^{2+2\alpha}+(\beta+o(1)e^{-\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}+e^{-2l\sin\frac{\pi}{k}})^2] \end{aligned} \tag{4.18}$$

Now we have

$$E_\varepsilon[u_0+\phi_l, v_0e^{\sqrt{-1}}\psi_l] = G(l) + k(1-\beta)^{\frac{3}{2}}I(w) + E[0, v_0],$$

where $k(1-\beta)^{\frac{3}{2}}I(w)+E[0,v_0]$ is independent on l. Hence the critical point of $E_\varepsilon[u_0+\phi_l, v_0e^{\sqrt{-1}\psi_l}]$ with respect to l is equivalent to the critical point of $G(l)$. In the following, we prove the existence of critical point of $G(l)$ and analyze the location of spikes, which depends on β.

Suppose $|\beta| = O(\varepsilon^p)$ with the power $p \geq 4+3\alpha$, then

$$\log\left(\frac{1}{\varepsilon}\right)\beta^2\left(\frac{1}{\varepsilon}\right)^{2+2\alpha} \ll \frac{\beta}{l^2}.$$

Since $1 \ll (\frac{1}{\varepsilon}-l) \ll \frac{1}{\varepsilon}$, we have

$$e^{-l\sin(\frac{\pi}{k})} \ll \frac{\beta}{l^2}.$$

We denote γ_1 and γ_3 to be two fixed constants with $\gamma_1 > \gamma_3 > 1+\frac{p}{2}$. When

$$l \in \left[\frac{1}{\varepsilon} - \gamma_1 \ln\frac{1}{\varepsilon}, \frac{1}{\varepsilon} - \gamma_3 \ln\frac{1}{\varepsilon}\right],$$

we have

$$e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)} \ll \frac{\beta}{l^2}.$$

We choose γ_2, γ_4 such that $0 < \gamma_2 < \gamma_4 < \frac{p}{2}+1$. When

$$l \in \left[\frac{1}{\varepsilon} - \gamma_4 \log\left(\frac{1}{\varepsilon}\right), \frac{1}{\varepsilon} - \gamma_2 \log\left(\frac{1}{\varepsilon}\right)\right],$$

we have

$$\frac{\beta}{l^2} \ll e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}.$$

Base on above analysis, we have

- Case 1

  When $l \in \left[\frac{1}{\varepsilon} - \gamma_1 \ln\frac{1}{\varepsilon}, \frac{1}{\varepsilon} - \gamma_3 \ln\frac{1}{\varepsilon}\right]$, we have

  $$G(l) = C_1\frac{\beta}{l^2}(1+o(1))$$

  and $G(l)$ strictly decreases with respect to l, since $\beta > 0$.

- Case 2

  When $l \in \left[\frac{1}{\varepsilon} - \gamma_4 \log\left(\frac{1}{\varepsilon}\right), \frac{1}{\varepsilon} - \gamma_2 \log\left(\frac{1}{\varepsilon}\right)\right]$, we have

$$G(l) = C_2 e^{-2\sqrt{1-\beta}(\frac{1}{\varepsilon}-l)}(1+o(1)) \tag{4.19}$$

  and $G(l)$ strictly increases .

Consequently, since $G(l)$ is bounded from below, there will be a local minimizer l_ε, which satisfies

$$\frac{1}{\varepsilon} - \gamma_3 \ln \frac{1}{\varepsilon} < l_\varepsilon < \frac{1}{\varepsilon} - \gamma_4 \ln \frac{1}{\varepsilon}$$

and γ_3, γ_4 can be very close to $\frac{p}{2}+1$.

We may suppose β to be even smaller, for example, $\beta = O(e^{-\left(\frac{1}{\varepsilon}\right)^q})$ with arbitrary positive constant $0 < q < 1$. Since when $\frac{1}{\varepsilon} - l \gg \left(\frac{1}{\varepsilon}\right)^q$, $C_1 \frac{\beta}{l^2}$ is the main part of $G(l)$, while $\frac{1}{\varepsilon} - l \ll \left(\frac{1}{\varepsilon}\right)^q$, $C_2 e^{-2(\frac{1}{\varepsilon}-l)}$ is the main part of $G(l)$, we can find a local minimizer with the same argument as above, but here the minimizer $l_\varepsilon \notin \Lambda$. In fact if we consider generally,

$$l \in \Lambda' = \left\{ l \middle| 1 \ll \frac{1}{\varepsilon} - l \ll \frac{1}{\varepsilon} \right\}$$

instead of $l \in \Lambda$, we can also follow the same argument as this thesis without any change. When $\beta = O(e^{-\left(\frac{1}{\varepsilon}\right)^q})$, it is easy to find a local minimizer $l_\varepsilon \in \Lambda'$.

Hence for $Ce^{-\left(\frac{1}{\varepsilon}\right)^q} \le \beta \le C\varepsilon^{4+3\alpha}$, where $0 < q < 1, 0 < \alpha < \frac{1}{2}$, we get $G(l)$ and further $E_\varepsilon[u_0 + \phi_l, v_0 e^{\sqrt{-1}}\psi_l]$ have a critical point l_ε (in fact, it is a local minimizer), then we have $c(l_\varepsilon) = 0$ in virtue of lemma 4.1. Hence we can get a solution $(u_\varepsilon,\ v_\varepsilon)$ of problem (1.4) with the form

$$u_\varepsilon(x) = \sum_{i=1}^{k} P_{\xi_i^\varepsilon} w(x) + \phi(x) = \sum_{i=1}^{k} w_\beta(x - \xi_i^\varepsilon) - \sum_{i=1}^{k} \Phi_{i,\varepsilon}(x) + \phi(x), \tag{4.20}$$

where

$$\xi_j^\varepsilon = l_\varepsilon e^{\sqrt{-1}\frac{2\pi(j-1)}{k}},\ j = 1, ..., k$$

and

$$v_\varepsilon(x) = S_\varepsilon(r) e^{\sqrt{-1}\theta} e^{\sqrt{-1}\psi(x)}, \quad \text{where } r = |x|,\ e^{\sqrt{-1}\theta} = \frac{x}{r}. \tag{4.21}$$

It is very easy to find that

$$\begin{cases} u_\varepsilon \in \Sigma_1 \Longleftrightarrow \phi \in \Sigma_1, \\ v_\varepsilon \in \Sigma_2 \Longleftrightarrow \varphi \in \Sigma_2, \end{cases}$$

So $(u_\varepsilon, v_\varepsilon) \in \Sigma$. Lemma 3.3 and inequality (2.10) show that

$$\phi(x),\ \psi(x) \text{ and } \Phi_{i,\varepsilon}(x) \ \to 0 \quad \text{uniformly in } C^0(B_\varepsilon). \tag{4.22}$$

We set $u(x) = u_\varepsilon(\frac{x}{\varepsilon})$ and $v(x) = v_\varepsilon(\frac{x}{\varepsilon})$. Then (u, v) solves problem (1.2). From (4.20), (4.21) and (4.22), it follows that

$$\begin{cases} u(x) = \sum_{j=1}^{k} w_\beta(\dfrac{x}{\varepsilon} - \xi_j^\varepsilon) + o(1), \\ v(x) = S_\varepsilon(\frac{r}{\varepsilon}) e^{\sqrt{-1}\,\theta + o(1)}, \quad \text{where } r = |x|,\ e^{\sqrt{-1}\theta} = \frac{x}{r}, \end{cases}$$

which would complete the proof of the main theorem.

Remark :

From above argument, we find that if β is getting smaller, l_ε is also getting smaller. Because $\varepsilon l_\varepsilon$ means the distance between the spikes and vortex in the unit ball, we know that if β is smaller, the spikes is closer to the vortex which is located at the center. This is consistent with what we have stated at the beginning that the positive sign of β may contribute inter-component repulsion between spikes and vortex. But totally, the spikes will be very close to the boundary as we suppose $\frac{1}{\varepsilon} - l_\varepsilon \ll \frac{1}{\varepsilon}$.

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